Flow of a Gas near a Solid Surface

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The state of a gas near a solid surface is investigated. It is assumed that at a sufficiently large distance from the surface the particle distribution function is of the Chapman-Enskog form. The half-range analysis previously employed for parallel plate geometries (symmetric problems) can be adapted to the single-plate problem. Although the mathematical analysis differs, the slip coefficients are essentially identical with those obtained from the parallel plate problem (e.g., Couette flow). Detailed calculations are presented for both hard sphere and Maxwellian molecules. The recent work of Bakanov and Deryagin for hard sphere molecules, which is based on incorrect approximations, is discussed.

1. Introduction

THE state of a gas near a physical boundary has been analyzed by microscopic methods based on the Boltzmann equation to clarify the role of the particle distribution function near the boundary and the magnitude of the slip coefficients. The standard problems of Couette (shear) flow or heat flow between parallel plates have been treated by methods due to Wang-Chang and Uhlenbeck[1] as well as Mott-Smith[2] by using full-range expansions, and by Gross and Ziering[3–6] using half-range expansions. The work of Jackson[7] and Willis,[8] furthermore, focused attention on logarithmic singularities near a physical boundary. All these methods have been compared fully in the forementioned literature.

With the exception of a paper by Wang-Chang and Uhlenbeck[9] and some recent work by Bakanov and Deryagin[10] based on the half-range method, the forementioned methods treated the symmetric parallel plate problems. The question of a shearing gas moving near a single plate, however, poses some special problems. Nonetheless, as will be shown below, the slip coefficients obtained are of similar magnitude to those predicted by the parallel plate geometry. The work of Bakanov and Deryagin[10] for the case of hard sphere molecules is based on some incorrect approximations that will be corrected below.

For the linear problem of a gas moving past a fixed wall and sustaining a velocity gradient as given by the Chapman-Enskog theory for shear flow, a mathematical analysis by kinetic theory will be given below. The related problem of a gas sustaining a temperature gradient and interacting with a surface has not been considered, as the two effects are decoupled in a linear treatment. (The extension, however, is not too difficult. See Refs. 5 and 6.) The application of microscopic techniques such as those just mentioned to the more realistic nonlinear problem, which is of immediate practical interest for high-speed, low-density flight, is exceedingly complex. It is all the more important, then, to use the linear analysis as a guideline, in order to put on a proper footing the necessary approximations that undoubtedly will have to be made in a nonlinear analysis. Thus, for instance, the importance and validity of specific boundary conditions and the relative merit of various moment approximations can be clarified in a linear theory. To a large extent this information may provide guidance only in an indirect sense, in that it will caution the investigator about certain methods and approximations. However, certain gross characteristics (such as slip coefficients) are obtainable from a microscopic linear analysis and may have a more general validity extending into the nonlinear domain. Another and more specific testing area for linear theories, however, is the effect of specific molecular laws of interaction on physical quantities of interest. This will be discussed in more detail in Sec. 5.

2. Shear Flow

The basic problem considered is that of a single infinite wall situated at $x = 0$ and interacting with a gas that, at sufficient distance from the wall, has a Chapman-Enskog distribution. The basic equation for the correction ϕ to the distribution function is (linearized Boltzmann equation; $f = (1 + \phi)$, where$f_0 \phi < 1$)

$$c_x(\partial\phi/\partial x) = J(\phi) \tag{1}$$

In accordance with the half-range procedure (see Refs. 3–5), write

$$\phi = \phi^+ \qquad c_x > 0$$

$$\phi = \phi^- \qquad c_x < 0$$

so that

$$\phi = \phi^+ \left\{\frac{1 + \text{sign}c_x}{2}\right\} + \phi^- \left\{\frac{1 - \text{sign}c_x}{2}\right\}$$

where $\text{sign}c_x = +1$ for $c_x > 0$, and $\text{sign}c_x = -1$ for $c_x < 0$. The microscopic boundary conditions then are

$$f^+(x = 0) = f_0$$

$$f^-(x = d) = f_0\{1 + 2c_zu_z(d) - lc_xc_z(\partial u_z/\partial x)(d)\} \tag{2}$$

where $\mathbf{c}$ is the nondimensional particle velocity $\mathbf{c} = \mathbf{v}/(2kT/m)^{1/2}$, and $\mathbf{u}$ is the nondimensional flow velocity $\mathbf{u} = \mathbf{q}/(2kT/m)^{1/2}$.

The authors have assumed for simplicity that all particles interacting with the wall are reflected diffusely. An extension allowing for both specular and diffuse reflection is straightforward.[3] At a sufficient distance from the wall ($x = d$), one assumes the characteristic Chapman-Enskog solution,[11] where l is proportional to the mean-free path, the constant of proportionality depending on the particular law of molecular interaction. Thus the forementioned boundary conditions for the correction terms are

$$\phi^+(x = 0) = 0$$

$$\phi^-(x = d) = 2c_zu_z(d) - lc_zc_x(\partial u_z/\partial x)(d) \tag{3}$$

In the four-function approximation for the shear flow problem of Refs. 4 and 5, write

$$\phi^\pm = a_0{}^\pm(x)c_z + a_1{}^\pm(x)c_zc_x \tag{4}$$

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so that

$$q_z = \frac{1}{4}\left(\frac{2kT}{m}\right)^{1/2}\left\{a_0^+ + a_0^- + \frac{a_1^+ - a_1^-}{\pi^{1/2}}\right\} \tag{5}$$

$$P_{xz} = \frac{nkT}{2\pi^{1/2}}\left\{a_0^+ - a_0^- + \frac{\pi^{1/2}}{2}(a_1^+ + a_1^-)\right\} \tag{6}$$

The half-range moment equations obtained by operating upon Eq. (1) respectively with

$$\int_{-\infty}^{\infty} c_z(1 \pm \operatorname{sign} c_x)e^{-c^2}\{\ \ldots\ \}\, d\mathbf{c}$$

and

$$\int_{-\infty}^{\infty} c_z c_x(1 \pm \operatorname{sign} c_x)e^{-c^2}\{\ \ldots\ \}\, d\mathbf{c}$$

are, on rearrangement,

$$(d/dx)a_0^{\pm} = (a_0^+ - a_0^-)A_1 + (a_0^+ + a_1^-)A_2 \pm (a_1^+ - a_1^-)A_3 \tag{7}$$

$$(d/dx)a_1^{\pm} = \pm(a_0^+ - a_0^-)B_1 \pm (a_1^+ + a_1^-)B_2 + (a_1^+ + a_1^-)B_3 \tag{8}$$

where

$$\begin{aligned} A_1 &= \frac{I_1 - (\pi^{1/2}/2)I_2}{\pi(1 - \pi/4)} & B_1 &= \frac{I_2 - (\pi^{1/2}/2)I_1}{\pi(1 - \pi/4)} \\ A_2 &= \frac{I_2 - (\pi^{1/2}/2)I_3}{\pi(1 - \pi/4)} & B_2 &= \frac{I_3 - (\pi^{1/2}/2)I_2}{\pi(1 - \pi/4)} \\ A_3 &= \frac{(\pi^{1/2}/2)I_4}{\pi(1 - \pi/4)} & B_3 &= \frac{I_4}{\pi(1 - \pi/4)} \end{aligned} \tag{9}$$

where the I_n's are collision integrals whose numerical value has been determined in Refs. 4 and 5 for hard sphere and Maxwellian molecules, respectively. The solutions of Eqs. (7) and (8) are of the standard form

$$a_i^{\pm} = \Sigma_j b_{ij}^{\pm} e^{\alpha j x}$$

The secular equation shows a doubly degenerate solution, so that the solution of Eqs. (7) and (8) is as follows:

$$a_i^{\pm} = c_i^{\pm} + d_i^{\pm}x + b_i^{\pm}e^{\alpha x} + g_i^{\pm}e^{-\alpha x} \tag{10}$$

Because of the boundary conditions at large $x(x \to d)$, the growing exponential solution is neglected, that is, $b_i^{\pm} = 0$. The eigenvalues and eigenvectors for the exponential part of the solution are identical with those previously obtained for the Couette flow (parallel plate) problem. The nonexponential solution is obtained easily by substitution of Eq. (10) in Eqs. (7) and (8). The boundary conditions are

$$\begin{aligned} a_0^+(x = 0) &= 0 & a_0^-(x = d) &= 2u_z(d) \\ a_1^+(x = 0) &= 0 & a_1^-(x = d) &= -l(\partial u_z/\partial x)(d) \end{aligned} \tag{11}$$

The case of Maxwellian molecules will be treated first, for reasons that will become apparent later.

3. Maxwellian Molecules

For Maxwellian molecules, $l = \frac{8}{3}\lambda$, where[11]

$$\lambda^{-1} = 2n\pi A_2(5)(K/kT)^{1/2}$$

Using the values of the collision integrals obtained in Ref. 5, it is found that the numerical values of the coefficients (9) are

$$\begin{aligned} A_1 &= -4.226/\lambda & B_1 &= 3.370/\lambda \\ A_2 &= -0.3750/\lambda = 1/l & B_2 &= 0 \\ A_3 &= -0.6938/\lambda & B_3 &= 0.7829/\lambda \end{aligned}$$

The solution matrix for Maxwellian molecules is

$$\begin{aligned} a_0^+ &= c_0 + 2A_2c_1x - 2.3453ge^{-\alpha x} \\ a_0^- &= c_0 + 2A_2c_1x - ge^{-\alpha x} \\ a_1^+ &= c_1 + 2.2414ge^{-\alpha x} \\ a_1^- &= c_1 - 0.7233ge^{-\alpha x} \end{aligned}$$

where $\alpha = 3.0582/\lambda$. Applying boundary conditions (11), it is found that

$$\begin{aligned} c_1 &= -l(\partial u_z/\partial x)(d) \\ c_0 &= 1.0462l(\partial u_z/\partial x)(d) \\ g &= 0.4461l(\partial u_z/\partial x)(d) \\ u_z(d) &= l(\partial u_z/\partial x)(d)\{0.5231 + (d/l)\} \end{aligned}$$

where terms of the order $e^{-\alpha d}$ have been neglected. From Eq. (5), one writes for the flow velocity

$$q_z(x) = \left(\frac{2kT}{m}\right)^{1/2} l\,\frac{\partial u_z}{\partial x}(d)\left\{0.5231 + \frac{x}{l} - 0.1865e^{-\alpha x}\right\}$$

and hence

$$\left.\frac{\partial q_z}{\partial x}\right|_{x=d} = \left(\frac{2kT}{m}\right)^{1/2}\frac{\partial u_z}{\partial x}(d)$$

Introducing the viscosity coefficient μ so that $P_{xz} = -\mu(\partial q_z/\partial x)|_{x=d}$, one finds, in agreement with the Chapman-Enskog solution,

$$P_{xz} = \frac{nkTc_1}{2} = -\rho\bar{v}\lambda\,\frac{\pi^{1/2}}{3}\left(\frac{2kT}{m}\right)^{1/2}\frac{\partial u_z}{\partial x}(d)$$

where $\bar{v} = (2/\pi^{1/2})(2kT/m)^{1/2}$, so that $\mu = (\pi^{1/2}/3)\rho\bar{v}\lambda$ and

$$q_z(x) = 1.1284\,\frac{2\mu}{\rho\bar{v}}\frac{\partial q_z}{\partial x}(d)\left\{1.0462 + \frac{2x}{l} - 0.3730e^{-\alpha x}\right\}$$

Thus at $x \to d$ one finds that the slip velocity is

$$q_c = (2\mu/\rho\bar{v})(\partial q_z/\partial x)(d)[1.1805]$$

which compares with Maxwell's slip coefficient:[12]

$$q_c^M = (2\mu/\rho\bar{v})(\partial q_z/\partial x)(d)$$

The slip coefficients determined by various techniques for the Couette flow problem for Maxwellian molecules are as follows:

Mott-Smith[2]		1.336
Wang-Chang and Uhlenbeck[9]	2nd approx.	1.383
	3rd approx.	1.126
	4th approx.	1.134
Welander[13]		1.210
Willis[14]		1.127
half-range[5]		1.160

Thus, with the exception of the Mott-Smith and the lowest Wang-Chang and Uhlenbeck approximation, these results are in fairly good agreement with the forementioned results for the single plate.

4. Hard Sphere Molecules

For molecules interacting as hard spheres, $l = 5\pi^{1/2}\lambda/4$, where $\lambda^{-1} = 2^{1/2}n\pi\sigma^2$. From Ref. 4, one substitutes for the collision integrals in Eq. (9) to obtain the following numerical coefficients for Eqs. (7) and (8):

$$\begin{aligned} A_1 &= -2.8938/\lambda & B_1 &= 2.1300/\lambda \\ A_2 &= -0.3719/\lambda & B_2 &= -0.0704/\lambda \\ A_3 &= -7.0126/\lambda & B_3 &= -7.9133/\lambda \end{aligned}$$

The most significant difference between these values and those obtained previously for Maxwellian molecules is that B_2 does not vanish; that is, for hard spheres, I_3 is not equal to $(\pi^{1/2}/2)I_2$. There is no a priori reason to suspect that this fortuitous relation in the case of Maxwellian molecules should be applicable universally to other laws of molecular interaction and in particular to hard sphere molecules. These integrals are determined completely and, for hard sphere molecules, can be evaluated analytically by a straightforward though tedious process. The previous calculations of the author have been checked and found to be accurate. The reasoning of Bakanov and Deryagin, who assume that $I_3 = (\pi^{1/2}/2)I_3$ also for hard sphere molecules, is wrong, as kinetic theory often is known to exhibit peculiar properties pertaining to a particular law of molecular interaction.

A further consequence of the fact that $B_2 \neq 0$ is evidenced in the solution below, in that c^+ is not equal to c^- (in contrast to the foregoing analysis for Maxwellian molecules). The contention raised by Bakanov and Deryagin that this always be so, even for the Couette flow problem,[4, 5] contradicts the basic equations (7) and (8). It is conceivable that in higher approximations this condition is satisfied more nearly for hard sphere molecules. For the Couette flow problem, however, their argument is wrong. That problem differs from the forementioned in that it had the inherent symmetry about the midplane as a consequence of the two plates moving in opposite directions with equal speed.[4, 5] Thus, halfway between the plates, the coefficients of the half-range distributions do indeed have the same numerical values, although these may differ in sign in accordance with these coefficients being attached to either odd or even moments. This is completely consistent with the forementioned symmetry condition. At no other point than halfway can these coefficients be expected to match, as the two streams have respectively traveled unequal distances from the respective plates and have undergone correspondingly different randomizations due to collisions.

This will be discussed in more detail in a later section; proceeding with the solution, one finds

$$a_0^+ = c^+ + (c^+ - c^-)[A_1B_2 - A_2B_1/B_2]x + 4.3833ge^{-\alpha x}$$

$$a_0^- = c^- - (c^+ - c^-)[A_1B_2 - A_2B_1/B_2]x - ge^{-\alpha x}$$

$$a_1^+ = -(c^+ - c^-)(B_1/2B_2) - 4.5481ge^{-\alpha x}$$

$$a_1^- = -(c^+ - c^-)(B_1/2B_2) - 1.5262ge^{-\alpha x}$$

Applying boundary conditions (11), one finds

$$c^+ = 0.9639\, l(\partial u_z/\partial x)(d)$$

$$c^- = 1.0300\, l(\partial u_z/\partial x)(d)$$

$$g = -0.2199\, l(\partial u_z/\partial x)(d)$$

$$u_z(d) = (\partial u_z/\partial x)(d)\{0.515 + 1.0357(d/l)\}$$

From Eq. (5), one writes for the flow velocity

$$q_z(x) = (2kT/m)^{1/2}l(\partial u_z/\partial x)(d)\{0.4985 + 1.0357(x/l) - 0.0923e^{-\alpha x}\}$$

and hence one finds that, unlike the previous analysis for Maxwellian molecules, neither the flow velocity nor its derivative matches to the prescribed values at $x = d$. That is,

$$q_z(d) = l(\partial u_z/\partial x)(d)\{0.4985 + 1.0357(d/l)\}(2kT/m)^{1/2}$$

and

$$(\partial q_z/\partial x)(d) = 1.0357(2kT/m)^{1/2}(\partial u_z/\partial x)(d)$$

This failure of the four-function approximation to duplicate the assumed Chapman-Enskog solution at $x = d$ exactly will be discussed below. The viscous shear determined from Eq. (6) is:

$$P_{xz} = (nkT/2\pi^{1/2})[c^+ - c^-]\{1 - (\pi^{1/2}/2)(B_1/B_2)\}$$
$$= -0.5090\rho\bar{v}\lambda(2kT/m)^{1/2}(\partial u_z/\partial x)(d)$$

But from the forementioned

$$(\partial q_z/\partial x)(d) = 1.0357(2kT^{1/2}/m)(\partial u_z/\partial x)(d)$$

and hence

$$P_{xz} = -0.4914\rho\bar{v}\lambda(\partial q_z/\partial x)(d)$$

which is in agreement with the Chapman-Enskog solution, where $\mu = 0.4914\rho\bar{v}\lambda$. Introducing the viscosity coefficient, the flow velocity now is written as follows:

$$q_z(x) = 1.1272(2\mu/\rho\bar{v})(\partial q_z/\partial x)(d)\{0.9626 + (2x/l) - 0.1782e^{-\alpha x}\}$$

Thus the slip velocity is $1.0850(2\mu/\rho\bar{v})(\partial q_z/\partial x)(d)$. The value for the slip velocity obtained by Bakanov and Deryagin is 1.09, whereas Mott-Smith[2] obtains 1.297 using a full-range expansion. The half-range expansion for the symmetric problem[4] yielded 1.065.

5. Discussion

The detailed description of a nonuniform gas interacting with a physical boundary has been of concern here. Possible numerical differences in the flow parameters arising out of 1) consideration of the single plate vs the double plate problem and 2) different molecular laws of force have been emphasized in this linearized analysis. Judging by the slip flow coefficients obtained in the forementioned analysis, the numerical differences are not significant in either case. Nonetheless, the analytical differences in the various solutions previously uncovered generally should caution an investigator against generalizations. It would be interesting to be able to compare exact solutions of the linearized single and double plate problems. In the framework of the foregoing formulation, however, the author expects that the single plate exact solution would differ from the double plate exact solution. The reason for this is that, for the single plate problem in the just mentioned formulation, the boundary condition away from the plate is at most a reasonable, macroscopically justifiable assumption; it is by no means on the same level of mathematical rigor as the microscopic boundary conditions in terms of distribution functions used for the other boundaries in the two problems.

Returning to the influence of molecular models on the physical quantities of interest, the author found for the case of hard spheres that the incorrect assumption of Bakanov and Deryagin [i.e., $I_3 = (\pi^{1/2}/2)I_2$] did not lead to a significantly different slip velocity when corrected. The ratio of two different collision integrals, however, for instance I_1/I_4, which yields[4] 7.376 for Maxwellian molecules and[5] 0.592 for hard sphere molecules, shows a significant variation and makes any predictions based on the results for Maxwellian molecules wholly inapplicable to hard sphere molecules.

As a general rule, generalizations of laws applicable to Maxwellian molecules should be exercised with care. It should be borne in mind that Maxwell invented, so to say, the inverse fifth-power law of molecular interaction as a mathematical convenience. The relative simplicity of the resulting collision integrals for this power law allowed Maxwell to construct the comparatively simple equations of transfer. The closely related fact that the Sonine-Legendre polynomials form a set of eigenfunctions for the collision operator in the case of Maxwellian molecules is well known. It is equally well known that neither of the forementioned simplifications applies to other molecular models, such as, for instance, rigid spheres.

Although, in principle, the eigenfunctions found for Maxwellian molecules can be used in connection with other molecular laws of force, off-diagonal elements occur which, although minor, nonetheless destroy the simplicity inherent in the treatment of Maxwellian molecules and introduce various corrections. This has been documented by Grad[15] and by Wang-Chang and Uhlenbeck.[1] The effect of these off-diagonal elements is illustrated clearly in the parallel plate shear and heat flow solution of Refs. 4–6. There it is found for Maxwellian molecules the degenerate solution ($a_i^{\pm} = c_i^{\pm} + d_i^{\pm}x$) shows that the constant coefficients ($c_i^{\pm}$) vanish for even velocity moments (in c_x), whereas the linear coefficients ($d_i^{\pm}$) vanish for odd velocity moments.[4] This is not the case, however, for hard sphere molecules.[5, 6]

The phenomenon of thermal diffusion presents an even more striking illustration. Thus, thermal diffusion effects vanish for an inverse fifth-molecular law of interaction only. Indeed, this particular phenomenon was not discovered by Maxwell simply because it is absent in the case of Maxwell molecules.

To return to the shear problem treated here, it can be shown that, because of the orthogonality of the eigenfunctions associated with the flow velocity (c_z) and viscous stress (c_zc_x) for Maxwellian molecules, there is one collision integral, namely I_3, which uniquely relates the two physical quantities, viz.,

$$P_{xz} = (c/I_3)(\partial q_z/\partial x)$$

where $-c/I_3$ is the viscosity coefficient μ. Higher velocity moment approximations do not contribute. For hard sphere molecules, however, although the equivalent low order velocity moment approximation very nearly determines the viscosity coefficient, there are small contributions from higher order approximations.

In view of these remarks, there appears to be little justification for Bakanov and Deryagin's inference. Starting with the Chapman-Enskog distribution, they then proceed to determine two collision integrals for hard sphere molecules by forcing the assumed distribution to conform to the Chapman-Enskog form. From the second-order treatment of Burnett[16] or the higher order treatments of Grad[15] (13 moments, or 20 moments), one would, of course, arrive at entirely different relationships. The treatment of Bakanov and Deryagin is not necessarily a bad approximation; it is, however, insufficient to justify a fortuitous relation peculiar to Maxwellian molecules as generally being valid for other molecular laws of force.

The extent to which the four-function half-range approximation employed here suffices to describe the shear flow phenomena can be judged by the numerical differences evident in the solution for hard sphere molecules between the flow quantities in the body of the gas and the postulated Chapman-Enskog type of solution at the boundary. The numerical differences are of the order of 3% for hard sphere molecules and do not occur for Maxwellian molecules.

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